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Fast and slow degrees of freedom coupling two different reservoirs

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We considered a Hamiltonian system that can be described by two generalized variables. One of them relaxes quickly when the system is in contact with a heat bath at fixed temperature, while the second one, the slow variable, mimics the interaction of the system with another heat bath at a lower temperature. The coupling between these variables leads to an energy flow between the heat baths. Allahverdyan and Nieuwenhuizen [Phys. Rev. E **62**, 845 (2000)] proposed a formalism to deal with such problem and calculated the steady states of the system and some related properties as entropy production, energy dissipation, etc. In this work we applied the formalism to a coupled system of ideal gases and also to an ideal gas interacting with a harmonic oscillator. If the temperatures of the heat baths are not too close, the Onsager relations do not apply.

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I. INTRODUCTION

Most processes occurring in nature are nonequilibrium ones and, as it is well known, there is no complete and detailed theory to account for these phenomena. On the other hand, statistical thermodynamics is a well established theory, which can be derived from the Gibbsian theory of the ensembles. The books of Callen [1] and Landau and Lifshitz [2] are fundamental references in the field of equilibrium statistical thermodynamics. In many situations we are more interested in processes rather than in a clear definition of states. Then, we very often ask for the rates of physical processes and what general properties can be derived without looking for the details of the system. The scope of nonequilibrium thermodynamics is ample and it is intended to describe physical, chemical, and biological processes. Notwithstanding the lack of a general theory, there are some formulations with emphasis in near equilibrium phenomena, the most notable of them is the classical reciprocity relations advanced by Onsager [3]. After the appearance of the Onsager reciprocity theorem, this field had a more consistent development, and nowadays some texts are also references in literature, like the books of de Groot and Mazur [4], and Prigogine [5].

The process we are interested in this work is the flow of energy between two heat baths at different temperatures. The flow of energy is mediated by a system described by two generalized coordinates. The interaction of the system with the heat bath at higher temperature T_1 is made through a fast variable x_1, while the heat bath at lower temperature T_2 interacts with the slow variable x_2. The coupling between the heat baths is accomplished by an interaction term involving the two variables x_1 and x_2. Allahverdyan and Nieuwenhuizen [6] formulated a very nice theory to treat this kind of problem when there is a large separation between the relaxation times of these variables. In fact, they applied their formalism to a pair of weakly interacting harmonic oscillators, showing the breakdown of the Onsager relations if T_1 is not close to T_2. An interesting result that arises from their general formalism is the proportionality between the entropy production and energy dissipation rates, to leading order on the ratio of the two relaxation times.

The purpose of this paper is twofold: first, we reviewed the formalism, stressing some points that seemed to us very important for a complete understanding of the method. Second, we applied the formalism to two simple systems. The simplest system we devised is the ideal gas, where the Hamiltonian contains only an interacting term between the two variables x_1 and x_2. Besides, we also considered a system, where the fast variable represents an ideal gas and the slow variable the harmonic oscillator. In both examples, the results we found are very similar to those found by Allahverdyan and Nieuwenhuizen for the two harmonic oscillators case. This paper is organized as follows: in Sec. II we revisit the general formalism proposed by Allahverdyan and Nieuwenhuizen (AN), detailing some steps of the derivation. Section III is devoted to some applications of the formalism. The cases for the interaction between two ideal gases, and the interaction between an ideal gas and a harmonic oscillator are explicitly considered. Finally, in Sec. IV, we present our conclusions.

II. THE FORMALISM

The system is represented by a general Hamiltonian $H(x_1,x_2)$, where x_1 and x_2 are two generalized stochastic variables. The system interacts with two different heat baths at fixed temperatures T_1 and T_2, where we take $T_2<T_1$. The interaction with the hotter reservoir is made through the fast variable x_1, while with the colder one is via the slow variable x_2. The indirect interaction between the reservoirs is implemented by adding a coupling term between the variables x_1 and x_2 into the Hamiltonian. A key parameter in the formalism is the ratio $\gamma=\Gamma_1/\Gamma_2$, where Γ_1 and Γ_2 are the relaxation times associated with the variables x_1 and x_2, respectively. In particular, we will focus our attention in cases where $\gamma\ll 1$. The formalism departs from the Langevin equation for the variables x_1 and x_2

$$\Gamma_i\dot{x}_i=-\frac{\partial}{\partial x_i}H(x_1,x_2)+\eta_i(t), \quad i=1,2, \tag{1}$$

where

$$\langle \eta_i(t)\eta_j(t')\rangle = 2\Gamma_i T_i \delta_{ij}\delta(t-t'), \quad i,j=1,2. \tag{2}$$

From Eqs. (1) and (2) we obtain the associated Fokker-Planck equation

$$\frac{\partial}{\partial t}P(x_1,x_2;t)+\sum_{i=1}^{2}\frac{\partial}{\partial x_i}J_i(x_1,x_2;t)=0, \tag{3}$$

$$J_i(x_1,x_2;t)=-\frac{1}{\Gamma_i}P(x_1,x_2;t)\frac{\partial}{\partial x_i}H(x_1,x_2) - \frac{T_i}{\Gamma_i}\frac{\partial}{\partial x_i}P(x_1,x_2;t), \tag{4}$$

where $P(x_1,x_2;t)$ is the probability distribution, J_1 and J_2 are the corresponding currents of probability. Following AN the stationary distribution to order γ^0 is given by

$$P_0(x_1,x_2)=P_0(x_2)P_0(x_1|x_2), \tag{5}$$

where

$$P_0(x_1|x_2)=\frac{1}{Z(x_2)}\exp[-\beta_1 H(x_1,x_2)],$$

$$Z(x_2)=\int dx_1 \exp[-\beta_1 H(x_1,x_2)], \tag{6}$$

$$P_0(x_2)=\frac{Z^{T_1/T_2}(x_2)}{\mathcal{Z}}, \quad \mathcal{Z}=\int dx_2 Z^{T_1/T_2}(x_2). \tag{7}$$

As usual $\beta_1=1/T_1$, where we are taking the Boltzmann constant equal to unity.

Assuming that Eq. (5) will remain valid for nonstationary states, that is,

$$P(x_1,x_2;t)=P(x_2;t)P(x_1|x_2;t), \tag{8}$$

we can write the following general definitions for the mean energy and entropy:

$$U=\int dx_1 dx_2 P(x_1,x_2;t)H(x_1,x_2), \tag{9}$$

$$S=-\int dx_1 dx_2 P(x_1,x_2;t)\ln P(x_1,x_2;t). \tag{10}$$

It is interesting to note that this expression for the entropy [7] can be written as $S=S_1+S_2$, where

$$S_1=\int dx_2 P(x_2;t)\left[-\int dx_1 P(x_1|x_2;t)\ln P(x_1|x_2;t)\right], \tag{11}$$

$$S_2=-\int dx_2 P(x_2;t)\ln P(x_2;t),$$

where S_1 is the entropy of the fast variable x_1, averaged over the slow variable x_2. S_2 is the entropy associated with the slow variable x_2. According to AN a free energy can be defined for this two-temperature system, namely, $F=-T_2\ln\mathcal{Z}$. This definition of F allows to generalize the usual expression for the free energy of a single heat bath

$$F=U-T_1S_1-T_2S_2, \tag{12}$$

where

$$\left(\frac{\partial F}{\partial T_1}\right)_{T_2}=-S_1, \quad \left(\frac{\partial F}{\partial T_2}\right)_{T_1}=-S_2. \tag{13}$$

For a system in contact with two different heat baths, there will be a constant flow of thermal energy at the steady states. As a consequence, there will be also a constant production of entropy and a dissipation of energy. As we are interested in computing these quantities, keeping terms beyond γ^0, we look for a steady state probability distribution in the following form:

$$P_1(x_1,x_2)=P_0(x_1,x_2)[1-\gamma A(x_1,x_2)]+O(\gamma^2), \tag{14}$$

where $A(x_1,x_2)$ must be determined. To accomplish this goal, we note that the stationary probability and its derivatives must vanish at the boundaries of the system. Also, the function $A(x_1,x_2)$ must satisfy the orthogonality condition $\int dx_1dx_2A(x_1,x_2)P_0(x_1,x_2)=0$. From the stationary condition for the probability, $(\partial/\partial t)P(x_1,x_2;t)=0$, we arrive at the following partial differential equation to order γ:

$$T_1\frac{\partial^2 A}{\partial x_1^2}+\left(\frac{\partial H}{\partial x_1}+2T_1\frac{\partial}{\partial x_1}\ln P_0\right)\frac{\partial A}{\partial x_1} = \frac{\partial^2 H}{\partial x_2^2}+\frac{\partial H}{\partial x_2}\frac{\partial}{\partial x_2}\ln P_0+\frac{T_2}{P_0}\frac{\partial^2 P_0}{\partial x_2^2}. \tag{15}$$

From this equation we obtain a general expression for the auxiliary function $A(x_1,x_2)$. However, this solution must also be consistent to order γ^2. This implies that the function $A(x_1,x_2)$ must also satisfy the eigenvalue equation

$$\left[\frac{\partial^2 H}{\partial x_2^2}+\frac{\partial H}{\partial x_2}\frac{\partial}{\partial x_2}+T_2\frac{\partial^2}{\partial x_2^2}\right](P_0A)=\lambda(P_0A). \tag{16}$$

Using Eqs. (4) and (14) we obtain the equations for the currents in the stationary state,

$$J_1(x_1,x_2)=\frac{T_1}{\Gamma_2}P_0(x_1,x_2)\frac{\partial}{\partial x_1}A(x_1,x_2), \tag{17}$$

$$\begin{aligned} J_2(x_1,x_2)=&\frac{(T_1-T_2)}{T_1\Gamma_2}P_0(x_1,x_2)[\delta F_2(x_1,x_2)] \\ &-\gamma P_0(x_1,x_2) \\ &\times\left[A(x_1,x_2)\frac{(T_1-T_2)}{T_1\Gamma_2}[\delta F_2(x_1,x_2)] \right.\\ &\left.-\frac{T_2}{\Gamma_2}\frac{\partial}{\partial x_2}A(x_1,x_2)\right]. \end{aligned} \tag{18}$$

As we seek expressions for the entropy production and energy dissipation up to terms of order γ^2, we kept in the expression for $J_2(x_1,x_2)$ a second order term in γ. The term $\delta F_2(x_1,x_2)$ that appears in the equation of $J_2(x_1,x_2)$, is defined by

$$\delta F_2(x_1,x_2) = -\frac{\partial}{\partial x_2}H(x_1,x_2) + \int dy P_0(y|x_2)\frac{\partial}{\partial x_2}H(y,x_2). \tag{19}$$

As pointed out by AN, this term represents the difference between the force acting on the subsystem governed by the slow variable x_2 and its mean value obtained by averaging over the fast variable x_1. The change of the total entropy, that is, the variation of the system entropy plus the variation of the entropies of both heat baths can be written, at the stationary state, as

$$\dot{S}_{Tot} = (\beta_1 - \beta_2)\dot{Q}_2, \tag{20}$$

where

$$\dot{Q}_2 = -\int dx_1 dx_2 H(x_1,x_2)\frac{\partial}{\partial x_2}J_2(x_1,x_2;t). \tag{21}$$

To obtain Eq. (20) we had to assume that the entropy and energy of the system are constant and that the heat baths are indeed in equilibrium.

By taking the time derivative of Eq. (12), using Eqs. (9)–(11) and employing the definitions of J_1 and J_2 as given by Eq. (4), we arrive at the following equation for the free energy rate:

$$\dot{F}(t) = (T_1 - T_2)\int dx_1 dx_2 J_2(x_1,x_2;t)\frac{\partial}{\partial x_2}\ln P(x_1|x_2;t) - \int dx_1 dx_2 P(x_1,x_2;t)\sum_{i=1}^{2}\frac{1}{\Gamma_i}\left[\frac{\partial}{\partial x_i}H(x_1,x_2) + T_i\frac{\partial}{\partial x_i}\ln P(x_1,x_2;t)\right]^2. \tag{22}$$

The preceding equation has a very nice interpretation. Writing the free energy rate as $\dot{F} = \dot{W} - \dot{\Pi}$, we can identify $\dot{W}$ with the work done to keep the system in the nonequilibrium state. On the other hand, the strictly positive term $\dot{\Pi}$ is the energy dissipation. Therefore, in the stationary state for which $\dot{F} = 0$, the work done must be equal to the dissipated energy.

We can get explicit expressions for the total entropy production rate $\dot{S}_{Tot}$ and energy dissipation rate $\dot{\Pi}$ at the steady state. In order to find $\dot{S}_{Tot}$, we start from Eq. (20), and use of Eqs. (21) and (18) gives

$$\dot{S}_{Tot} = \gamma\frac{\kappa^2}{T_2\Gamma_1}\langle(\delta F_2)^2\rangle_1 + \gamma^2\frac{\kappa}{\Gamma_1}\left\langle(\delta F_2)\frac{\partial A}{\partial x_2}\right\rangle_0, \tag{23}$$

where $\kappa = (T_1 - T_2)/T_1$, and the labels $0(1)$ in the brackets, mean averages with respect to distributions $P_{0(1)}$. In the case of $\dot{\Pi}$, we start from the second term on the right-hand side of Eq. (22), which can be reduced with the help of Eq. (4) to

$$\dot{\Pi} = \int dx_1 dx_2 \frac{1}{P_1(x_1,x_2)}(\Gamma_1 J_1^2 + \Gamma_2 J_2^2). \tag{24}$$

Finally, introducing in the last expression, Eqs. (17) and (18) we find, after some algebraic manipulations, an expression for $\dot{\Pi}$ that reads

$$\dot{\Pi} = \gamma\frac{\kappa^2}{\Gamma_1}\langle(\delta F_2)^2\rangle_1 + \gamma^2\frac{\kappa(2T_2 - T_1)}{\Gamma_1}\left\langle(\delta F_2)\frac{\partial A}{\partial x_2}\right\rangle_0. \tag{25}$$

The most important result is the general linear relation observed between $\dot{\Pi}$ and $\dot{S}_{Tot}$ to the leading order in γ, namely,

$$\dot{\Pi} = T_2\dot{S}_{Tot} + O(\gamma^2), \tag{26}$$

where only the temperature of the colder heat bath appears explicitly.

III. APPLICATIONS

In what follows we will be considering three examples of application of the formalism presented in the preceding section.

A. Two harmonic oscillators

The interaction of an harmonic oscillator with a heat bath is a well studied problem. For instance, Ford, Kac, and Mazur [8] have shown that a particle coupled harmonically to the bath and subjected to an arbitrary force to a fixed center exhibits a Brownian motion. They considered the heat bath as a chain of coupled harmonic oscillators, and the related Langevin equation is a contracted description, where the bath is described only by two parameters, the temperature and the damping constant. Also, Ullersma [9] consider a similar system, where the oscillator is linearly coupled with the oscillators that model the heat bath. He found an exact solution for this system when the oscillators modeling the bath are not coupled themselves. The same results were also derived by Caldeira and Legget [10] based on the application of the influence-functional method of Feynman and Vernon to a quantum version of the problem studied by Ullersma. In the limit where the Planck constant goes to zero, their results reduce to the classical Fokker-Planck equation.

Here we consider the same model studied by AN. The Hamiltonian for the model system is

$$H = \tfrac{1}{2}ax_1^2 + \tfrac{1}{2}ax_2^2 + gx_1^2x_2^2, \tag{27}$$

where a and g are positive constants. The system is represented by a pair of weakly interacting oscillators, and we keep only first order terms in g. This Hamiltonian model is very symmetrical, and this fact will enable us to determine with a little effort the properties of interest. Following the formalism outlined in the preceding section, the distribution $P_0(x_1,x_2)$ is

$$P_0(x_1,x_2)=\left(\frac{\beta_1\beta_2 a^2}{4\pi^2}\right)^{1/2}\exp\left(-\frac{\beta_1 a x_1^2}{2}-\frac{\beta_2 a x_2^2}{2}\right)\times\left(1+\frac{gT_1}{a^2}-\frac{gT_1\beta_2 x_2^2}{a}+\frac{g x_2^2}{a}-g\beta_1 x_1^2 x_2^2\right). \tag{28}$$

The differential equation for $A(x_1,x_2)$ is given by

$$T_1\frac{\partial^2 A}{\partial x_1^2}-(a x_1+2g x_1 x_2^2)\frac{\partial A}{\partial x_1}=2g(T_1-T_2)\left(\beta_1 x_1^2+\beta_2 x_2^2-\beta_1\beta_2 a x_1^2 x_2^2-\frac{1}{a}\right). \tag{29}$$

The general solution for this equation is

$$A(x_1,x_2)=gc(x_2)+g\frac{\kappa}{a}(\beta_2 a x_1^2 x_2^2-x_1^2), \tag{30}$$

where $c(x_2)$ is an arbitrary function of x_2. To guarantee consistency to order γ^2, Eq. (16) reduces to

$$T_2\frac{\partial^2 A}{\partial x_2^2}-a x_2\frac{\partial A}{\partial x_2}=\lambda A. \tag{31}$$

Expanding $c(x_2)$ in a power series of x_2 we find that $\lambda=-2a$, and $c(x_2)=\alpha(1-\beta_2 a x_2^2)$. The expression for $A(x_1,x_2)$ becomes

$$A(x_1,x_2)=g\alpha-g\alpha\beta_2 a x_2^2+g\frac{\kappa}{a}(\beta_2 a x_1^2 x_2^2-x_1^2). \tag{32}$$

This solution automatically satisfies the orthogonality condition. An elegant manner to find the yet undetermined constant α is to explore the symmetries of the differential equation for $A(x_1,x_2)$, Eq. (28). Observing that $A(x_2,x_1;\beta_2,\beta_1)=-A(x_1,x_2;\beta_1,\beta_2)$, we find that $\alpha=(T_1-T_2)/a^2$. Then, the final expression for $A(x_1,x_2)$ is

$$A(x_1,x_2)=\frac{g(T_1-T_2)}{a^2}(1-\beta_1 a x_1^2)(1-\beta_2 a x_2^2). \tag{33}$$

Once we have determined $A(x_1,x_2)$ the entropy production and energy dissipation calculations follow straightforward.

B. Two ideal gases

Now let us consider another example of application of the previous formalism. The Hamiltonian model for the system is taken as

$$H=g(x_1-x_2)^2. \tag{34}$$

This system can be thought as being formed by two ideal gases. One of them, described by a fast variable x_1, is in contact with the heat bath at temperature T_1, while the other, described by the slow variable x_2, is in contact with the colder heat bath at temperature T_2. We can think of these two gases as having very different masses, and the parameter γ can be related to the ratio between the masses of the light and heavy gases [11]. The gases are confined to a limited region of space between $-L/2<x_{1,2}<L/2$, and we have assumed a weak interaction between them, as given by Eq. (34). As in the case of two coupled harmonic oscillators, only configurational variables are considered.

In this case, we obtain the following expression for $P_0(x_1,x_2)$:

$$P_0(x_1,x_2)=\frac{1}{L^2}\left[1+g\frac{L^2}{12}(\beta_1+\beta_2)-g\beta_1 x_1^2-g\beta_2 x_2^2+2g\beta_1 x_1 x_2\right]. \tag{35}$$

The differential equation for $A(x_1,x_2)$ is given by

$$T_1\frac{\partial^2 A}{\partial x_1^2}-2g(x_1-x_2)\frac{\partial A}{\partial x_1}=0. \tag{36}$$

Going along the same steps developed for the case of two harmonic oscillators, that is, finding the general solution to first order in g, imposing consistency up to order γ^2 and applying the orthogonality condition we obtain a very simple solution for $A(x_1,x_2)$, which can be written as

$$A(x_1,x_2)=g(\beta_1-\beta_2)x_1x_2=-g\frac{\kappa}{T_2}x_1x_2. \tag{37}$$

With this expression for $A(x_1,x_2)$ it is easy to calculate the entropy production and energy dissipation rates. From Eqs. (23) and (25), we find

$$\dot{S}_{Tot}=\frac{g^2L^2}{3\Gamma_1}\frac{(T_1-T_2)^2}{T_1^2T_2}\left(\gamma-\frac{\gamma^2}{2}\right), \tag{38}$$

$$\dot{\Pi}=\frac{g^2L^2}{3\Gamma_1}\frac{(T_1-T_2)^2}{T_1^2}\left[\gamma-\gamma^2\frac{(2T_2-T_1)}{2T_2}\right]. \tag{39}$$

In this problem where we are considering heat transfer between two reservoirs, we can test the validity of the Onsager relations. These relations were derived originally by Onsager [3] for the linear regime. In our case this would correspond to a situation where $T_1\approx T_2$. If the heat bath temperatures are not too close, a general expression for the Onsager relation can be employed [6] to the nonlinear regime, that is,

$$\frac{\partial\dot{Q}_2}{\partial\beta_1}=\frac{\partial\dot{Q}_1}{\partial\beta_2}. \tag{40}$$

From Eqs. (20) and (23) we can write the following expression to order γ:

$$\frac{\partial \dot{Q}_2}{\partial \beta_1}-\frac{\partial \dot{Q}_1}{\partial \beta_2}=\gamma\frac{T_2\kappa}{\Gamma_1}\langle(\delta F_2)^2\rangle_1 +\gamma\frac{T_2}{\Gamma_1}(\beta_1-\beta_2)\left(\frac{\partial}{\partial \beta_1}+\frac{\partial}{\partial \beta_2}\right)\langle(\delta F_2)^2\rangle_1 . \tag{41}$$

If $\langle(\delta F_2)^2\rangle_1=\beta_2 f(\beta_1-\beta_2)$, where f is any positive function, the Onsager relation, as expressed by Eq. (40), is automatically satisfied, even in the nonlinear regime. Particularly, for the Hamiltonian model we are considering, given by Eq. (34), we can show that $\langle(\delta F_2)^2\rangle_1=g^2L^2/3$. Therefore, the Onsager relation breaks down in the nonlinear case. For this model, it is easy to show that

$$\frac{\partial \dot{Q}_2}{\partial \beta_1}-\frac{\partial \dot{Q}_1}{\partial \beta_2}=\gamma\frac{g^2L^2}{3\Gamma_1}\frac{T_2}{T_1}(T_1-T_2). \tag{42}$$

We would like to stress that even for this very simple model of two weakly interacting ideal gases the Onsager relations are violated.

C. Ideal gas and harmonic oscillator

As a final example, we consider a system formed by an oscillator interacting with an ideal gas. The Hamiltonian for this system is chosen to have the following form

$$H=\tfrac{1}{2}ax_2^2+g(x_1-x_2)^2, \tag{43}$$

where the fast variable x_1 is in contact with the heat bath at temperature T_1, and this variable is limited to the region $-L/2<x_1<L/2$. The slow variable x_2, describes an harmonic oscillator in contact with the heat bath at the lower temperature T_2. As usual, this variable is unbounded. In order to take into account a heat flow between reservoirs, a small coupling term was introduced into the Hamiltonian. As before, only linear terms in g will be allowed. Following the steps of Sec. II, first we determine the probability distribution function $P_0(x_1,x_2)$, that is,

$$P_0(x_1,x_2)=\left(\frac{\beta_2 a}{2\pi L^2}\right)^{1/2}\exp\left(-\frac{\beta_2 a x_2^2}{2}\right)\left(1+\frac{g\beta_1L^2}{12}+\frac{g}{a} -g\beta_1x_1^2-g\beta_2x_2^2+2g\beta_1x_1x_2\right). \tag{44}$$

The partial differential equation for $A(x_1,x_2)$ reads

$$T_1\frac{\partial^2 A}{\partial x_1^2}-2g(x_1-x_2)\frac{\partial A}{\partial x_1}=2g\kappa\beta_2 a x_1x_2 . \tag{45}$$

A solution that satisfies the boundary conditions, the eigenvalue equation for consistency to order γ^2 and the orthogonality condition can be written as

$$A(x_1,x_2)=g\kappa\frac{\beta_1\beta_2}{3}ax_1^3x_2-g\kappa\frac{\beta_1\beta_2}{4}aL^2x_1x_2 . \tag{46}$$

The corresponding equations for the entropy production and energy dissipation rates are given by

$$\dot{S}_{Tot}=\frac{g^2L^2}{3\Gamma_1}\frac{(T_1-T_2)^2}{T_1^2T_2}\left(\gamma-\gamma^2\frac{aL^2}{10T_1}\right), \tag{47}$$

$$\dot{\Pi}=\frac{g^2L^2}{3\Gamma_1}\frac{(T_1-T_2)^2}{T_1^2}\left[\gamma-\gamma^2\frac{aL^2}{10}\frac{(2T_2-T_1)}{T_1T_2}\right]. \tag{48}$$

As before, this simple interacting system also leads to the violation of the general Onsager relations in the nonlinear limit, that is, we have

$$\frac{\partial \dot{Q}_2}{\partial \beta_1}-\frac{\partial \dot{Q}_1}{\partial \beta_2}=\gamma\frac{g^2L^2}{3\Gamma_1}\frac{T_2(T_1-T_2)}{T_1}+O(\gamma^2). \tag{49}$$

IV. CONCLUSIONS

In this work we have analyzed a class of systems far from equilibrium based on the formalism developed by Allahverdyan and Nieuwenhuizen. They are stochastic systems in contact with two reservoirs at different temperatures. We have considered only the steady adiabatic states of the systems. One of the distinguishing features of these systems is that they present very different time scales for each one of its degrees of freedom. When the difference of temperatures of the two heat baths are large the systems are far from equilibrium. Despite this, a non-Gibbsian stationary probability distribution can be associated to the systems due to the huge difference between the relaxation times found in the studied systems. In the case of two variables, a thermodynamic description is found by a perturbative expansion on the ratio between the relaxation times of the fast and slow variables. We have revisited the work of Allahverdyan and Nieuwenhuizen, emphasizing some aspects of their formalism that seemed important to us. We also considered two different examples of application, involving steady state situations of some simple coupled Hamiltonian models. In these models we computed the energy dissipation and entropy production rates. We also verified the breakdown of the nonlinear Onsager relations for heat transfer occurring between the heat baths. Even for the case in which the temperatures of the two reservoirs are very close we have a violation of the Onsager relation, because for the models considered the products $\gamma\times g^2$ are very small and appear in both sides of Eq. (41). The formalism developed by these authors is interesting and we believe it can be extended to more complex interacting systems.

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